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Probabilistic Estimates for the Two-Dimensional Stochastic Navier–Stokes Equations

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We consider the Navier–Stokes equation on a two-dimensional torus with a random force, white noise in time, and analytic in space, for arbitrary Reynolds number R. We prove probabilistic estimates for the long-time behavior of the solutions that imply bounds for the dissipation scale and energy spectrum as $R \to \infty$.

KEY WORDS: Two dimensional turbulence; enstrophy; dissipation scale.

1. INTRODUCTION

In two dimensions global existence and uniqueness of solutions of the Navier–Stokes equation is known for a large class of initial conditions and forcing, deterministic and random. In particular, for a bounded domain the unforced system has a finite dimensional attractor[1,2] and this persists for a bounded finite dimensional force.

In this paper we consider the Navier–Stokes equation with a random force, white noise in time and large scale in space and prove probabilistic estimates for the long time behaviour of the solutions. Our analysis is inspired by the recent paper by Mattingly and Sinai[6] who gave a conceptually simple proof of analyticity of the solutions of the 2D Navier–Stokes equation. We extend their analysis to the random case.

We consider the stochastic Navier–Stokes equation for the velocity field $\mathbf{U}(t, \mathbf{x})$ defined on the torus $\mathbf{T}_L = (\mathbf{R}/2\pi L\mathbf{Z})^2$:

$$d\mathbf{U} + ((\mathbf{U} \cdot \nabla)\, \mathbf{U} - \nu \nabla^2 \mathbf{U} + \nabla p)\, dt = d\mathbf{F} \tag{1}$$

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where $\mathbf{F}(t, \mathbf{x})$ is a Wiener process with covariance

$$EF_\alpha(s, \mathbf{x})\, F_\beta(t, \mathbf{y}) = \min\{s, t\}\; C_{\alpha\beta}\left(\frac{\mathbf{x}-\mathbf{y}}{L}\right) \tag{2}$$

and $C_{\alpha\beta}$ is a smooth function defined on the unit torus and satisfying $\partial_\alpha C_{\alpha\beta} = 0$. Equation (2) represents large scale forcing, the scale being the size of the box. Equation (1) is supplemented with the incompressibility condition $\nabla \cdot \mathbf{U} = 0 = \nabla \cdot \mathbf{F}$ and we will also assume the vanishing averages over the torus: $\int_{\mathbf{T}_L} \mathbf{U}(0, \mathbf{x}) = 0 = \int_{\mathbf{T}_L} \mathbf{F}(t, \mathbf{x})$ which imply that $\int_{\mathbf{T}_L} \mathbf{U}(t, \mathbf{x}) = 0$ for all times t.

(1) implies the transport equation for the vorticity $\Omega = \partial_1 U_2 - \partial_2 U_1$:

$$d\Omega + ((\mathbf{U} \cdot \nabla)\, \Omega - \nu \nabla^2 \Omega)\, dt = dG \tag{3}$$

where $G = \partial_1 F_2 - \partial_2 F_1$ has the covariance

$$EG(t, \mathbf{x})\, G(s, \mathbf{y}) = L^{-2} \min\{s, t\}\; \Gamma\left(\frac{\mathbf{x}-\mathbf{y}}{L}\right)$$

with $\Gamma = -\Delta \operatorname{tr} C$.

It is convenient to change to dimensionless variables s.t. ν and L become one. This is achieved by setting

$$\mathbf{U}(t, \mathbf{x}) = \frac{\nu}{L}\, \mathbf{u}\left(\frac{\nu}{L^2}\, t, \frac{1}{L}\, \mathbf{x}\right), \qquad \Omega(t, \mathbf{x}) = \frac{\nu}{L^2}\, \omega\left(\frac{\nu}{L^2}\, t, \frac{1}{L}\, \mathbf{x}\right)$$

Then $\mathbf{u}$ and ω live on the unit torus and satisfy (1) and (3) with ν and L replaced by 1, and C and Γ replaced by

$$c = \frac{L^4}{\nu^3}\, C, \qquad \gamma = \frac{L^4}{\nu^3}\, \Gamma$$

Going to the Fourier transform $\omega_{\mathbf{k}}(t) = (2\pi)^{-2} \int_{\mathbf{T}_1} e^{i\mathbf{k} \cdot \mathbf{x}} \omega(t, \mathbf{x})\, d\mathbf{x}$ with $\mathbf{k} \in \mathbf{Z}^2$ we may write the enstrophy equation as

$$d\omega_{\mathbf{k}} = \left(-\mathbf{k}^2 \omega_{\mathbf{k}} + \sum_{\boldsymbol{l} \in \mathbf{Z}^2 \setminus \{\mathbf{0}, \mathbf{k}\}} (\mathbf{k} \times \boldsymbol{l})\, |\boldsymbol{l}|^{-2}\, \omega_{\mathbf{k}-\boldsymbol{l}} \omega_{\boldsymbol{l}} \right) dt + df_{\mathbf{k}} \tag{4}$$

where $\mathbf{k} \times \boldsymbol{l} = k_1 l_2 - l_1 k_2$ and $\{f_{\mathbf{k}}\}$ are Brownian motions with $\bar{f}_{\mathbf{k}} = f_{-\mathbf{k}}$ and

$$Ef_{\mathbf{k}}(s)\, f_{\boldsymbol{l}}(t) = \min\{s, t\}\; \delta_{\mathbf{k}, -\boldsymbol{l}} \gamma_{\mathbf{k}}$$

and we have used the relation $\mathbf{u}_{\mathbf{k}} = i[(-k_2, k_1)/\mathbf{k}^2]\, \omega_{\mathbf{k}}$.

The dimensionless control parameter is the ω injection rate,

$$R = \frac{1}{2} \sum_{\mathbf{k} \in \mathbf{Z}^2} \gamma_{\mathbf{k}} = \frac{1}{2}\,\gamma(0) = \frac{1}{2}\,\frac{L^4}{\nu^3}\,\Gamma(0)$$

which is proportional to the third power of the Reynolds number $\mathrm{Re} = L^{4/3}\varepsilon^{1/3}\nu^{-1}$ ($\varepsilon = \frac{1}{2}\operatorname{tr} C(0)$ is the energy injection rate) in our model. We will be interested in the turbulent region $R \to \infty$; therefore, for convenience, we will always assume below that $R > 1$. We make the following assumption on the noise covariance:

$$\gamma_{\mathbf{k}} \leqslant C\,\mathrm{Re}^{-|\mathbf{k}|} \tag{5}$$

The coefficient of $|\mathbf{k}|$ could be different from 1, but we require exponential decay. The physically relevant case is the one with $\gamma_{\mathbf{k}} \neq 0$ only for a finite number of $\mathbf{k}$ with $|\mathbf{k}|$ of the order of unity.

To state our main result, define the enstrophy

$$\Phi = \tfrac{1}{2} \sum_{\mathbf{k}} |\omega_{\mathbf{k}}|^2 \tag{6}$$

and fix numbers $r > 1$, $\alpha > 1 + r$. Consider, for positive D, the norm

$$\|\omega\|_D = \sup_{\mathbf{k}}\, |\omega_{\mathbf{k}}|\, |\mathbf{k}|^r\, e^{D^{-\alpha} |\mathbf{k}|} \tag{7}$$

D will vary below, but r and α are fixed. The factor $|\mathbf{k}|^r$ is useful technically (and was already used in ref. 6).

Theorem. Let $\|\omega(0)\|_{D_0} \leqslant D_0^{\alpha} < \infty$ and $\Phi(0) = K < \infty$. Then, there exists a random function D_t, $D_t < \infty$ for all t, such that with probability 1, $\|\omega(t)\|_{D_t} < D_t^{\alpha}$. For any $t > C(\log D_0 + \log K)$, and for $D^2 > CR \log R$, with $R > 1$,

$$\mathrm{Prob}\{\, \|\omega(t)\|_D \leqslant D^{\alpha} \;\&\; \Phi(t) \leqslant D^2 \} \geqslant 1 - C e^{-c(D^2/R)} \tag{8}$$

Remark. Here and below, C (and c) are sufficiently large (small) constants, which may vary from place to place but that are uniformly bounded as $R \to \infty$. The theorem says that with probability one $\omega(t, \mathbf{x})$ is analytic for all times, the dissipation scale is (up to a logarithm) $> R^{-(1/2)\,\alpha}$ and the energy spectrum

$$e(k) \equiv k^{-1} \int_{S^1} d\hat{\mathbf{k}}\; E\, |\omega_{\hat{\mathbf{k}}k}|^2 \leqslant C R^{\tilde{\alpha}} k^{-(2r+1)}$$

with $k = |\mathbf{k}|$, where r can be taken arbitrary close to 1 and $\tilde{\alpha}$ arbitrary close to $1+r$. These bounds hold for any fixed time and also for the average of these quantities over any fixed time interval. For example, using Jensen's and Chebyshev's inequalities, one derives from (8)

$$\text{Prob}\left\{\frac{1}{T}\int_t^{t+T} |\omega_{\mathbf{k}}(s)|^2 \, ds > D^{2\alpha} k^{-2r} e^{-2D^{-\alpha}k}\right\} \leqslant Ce^{-c(D^2/R)}$$

Equation (8) also implies that all correlation functions of the type

$$E \prod_i \nabla^{n_i} \omega(t, x_i)$$

exist. This entered as an assumption in ref. 11, and was used there to derive some physical consequences concerning 2D turbulence.

Let us close this section with two comments. The first concerns the relationship of our model to the standard 2D turbulence picture.[3, 4] One considers (1) in infinite volume with the forcing as we do at spatial scale L, but *not periodic*, rather, for instance, having a smooth Fourier transform with compact support around L^{-1}. Then it is expected that a stationary state for Ω emerges for which the energy spectrum $e(k) = k^{-1} \int_{S^1} d\hat{\mathbf{k}} \int d\mathbf{x} \times e^{ik\mathbf{k}\cdot\mathbf{x}} E\Omega(\mathbf{x})\, \Omega(0)$ has two scaling regimes

$$e(k) \propto \begin{cases} k^{-3} & \eta^{-1} \gg k \gg L^{-1} \\ k^{-5/3} & k \ll L^{-1} \end{cases} \tag{9}$$

refered to as the direct (enstrophy) cascade regime and the inverse (energy) cascade regime respectively. The scale η is the "viscous scale" beyond which the $e(k)$ decays more rapidly and it scales like $\nu^{1/2}$. In particular, the total energy density $\int_0^\infty e(k)\, dk$ is infinite in the stationary state. This means that starting with say vanishing $\mathbf{u}$ at time zero, the energy density increases linearly with time and for the ensuing stationary state only the vorticity remains a well defined random field. One can also work in finite volume like in this paper by forcing the system in an intermediate scale $\eta \ll \ell \ll L$, provided the energy is absorbed by friction acting on the $|\mathbf{k}| \sim L^{-1}$ regime. This indeed is what one does in experimental[8] and numerical[9] approaches.

In our case the absence of the friction forces the energy to dissipate in the short scales too and the spectrum should be different from (9). Our bound above is certainly far from realistic, but one would expect the $e(k)$ to diverge as $R \to \infty$. It would be very interesting to get hold of the direct and inverse cascade regimes, but certainly much more sophisticated ideas are needed than what are used in the present paper.

The second comment concerns the uniqueness of the stationary state (the existence is standard and follows from compactness and Lemma 1 below). In the case of Gaussian noise like as we have there are two kinds of results in the literature regarding uniqueness. In ref. 5 one proves uniqueness, provided the noise is taken big enough in the ultraviolet, i.e., the $\gamma_{\mathbf{k}}$ are taken to have a lower bound $k^{-\alpha}$ for α sufficiently small. This assures that ergodicity results from the action of the noise. However, such a noise is not what one is interested in the turbulence problem. The second result[7] is for a smooth noise but viscosity large enough, i.e., in the nonturbulent regime. Then the Laplacean is the dominant term in Eq. (1) and the past is forgotten exponentially fast due to the viscous damping.

In the turbulent regime of large R, the number N of modes $\omega_{\mathbf{k}}$ that are not explicitely damped by viscosity goes to infinity as $R \to \infty$ (we get an upper bound CR^{α} for N). Nevertheless, in the absence of noise, the enstrophy and thus $\omega(t)$ tends to zero and this dissipativity should lead to uniqueness of the stationary state provided the noise is nonvanishing for these N modes. In the case of bounded noise, kicked at discrete times, uniqueness has been recently proven by Kuksin and Shirikyan.[10] For a uniqueness result in the case of Gaussian noise, analytic in space, also kicked at discrete times, see ref. 12.

From the physical point of view, the rate of convergence to the stationary state that could be obtained solely due to the effect of the noise would not be realistic. As $R \to \infty$, the relaxation time due to this mechanism would grow faster than any power of R while in actual fact relaxation to stationarity should be due to the nonlinearity and should be much faster.

2. TRANSITION PROBABILITIES

Define the region

$$U_D = \{\omega \mid \|\omega\|_D \leqslant D^{\alpha} \text{ and } \Phi \leqslant D^2\} \tag{10}$$

Then the basic proposition is

Proposition. Suppose $\omega(0) \in U_D$. Then there are positive constants A and a, independent of R, such that

$$\mathrm{Prob}\{\omega(t) \in U_{\sqrt{2e^{-t}}D}, \forall t, 0 \leqslant t \leqslant 1\} \geqslant 1 - AR^{2\alpha} e^{-a(D^2/R)} \tag{11}$$

It has a rather immediate

Corollary. Suppose $\omega(0) \in U_D$ and $D' > D$. Then

$$\text{Prob}\{\omega(t) \notin U_{\sqrt{2e^{-t}}\,D'}, \text{ for some } t \in [0, 1]\} \leqslant AR^{2\alpha} e^{-a(D'^2/R)} \tag{12}$$

Proof. Note that for $D < D'$, $U_D \subset U_{D'}$. Thus $\omega(0) \in U_{D'}$. Now the proposition implies the claim. ∎

Proof of the Theorem. Consider the Markov chain with transition probabilities

$$p(\omega, U) = \text{Prob}\{\omega(1) \in U \mid \omega(0) = \omega\} \tag{13}$$

Let, for $n \geqslant 0$, $U_n = U_{D_n}$ where $D_n^2 = 2a^{-1}R(\frac{1}{2}e)^n$ and define

$$p_{m,n} = \sup_{\omega \in U_m} p(\omega, U_n^c) \tag{14}$$

Since, by definition, $\sqrt{2e^{-1}}\, D_m \leqslant D_n$, for $m \leqslant n+1$, the corollary implies

$$p_{m,n} \leqslant A' e^{-((1/2)\,e)^n} \equiv A' \pi_n \tag{15}$$

for $m \leqslant n+1$ and $D_n^2 > CR \log R$ (so that $R^{2\alpha} e^{-a(D_n^2/2R)} \leqslant C'$, and we can take $A' = C'A$).

By assumption, $\omega(0) \in U_N$ for any $N < \infty$ such that D_0 in the theorem is less than D_N. Let $p_n(t) \equiv \text{Prob}\{\omega(t) \in U_n^c\}$. Then

$$p_n(t+1) \leqslant \text{Prob}\{\omega(t) \in U_{n+1}\}\; p_{n+1,n} + p_{n+1}(t) \leqslant p_{n+1,n} + p_{n+1}(t) \tag{16}$$

Suppose, inductively in $t \in \mathbf{N}$, that

$$p_n(t) \leqslant B\pi_n \tag{17}$$

for $n \geqslant N - t$. Then, for $n \geqslant N - t - 1$, (16), (15) and (17) yield

$$p_n(t+1) \leqslant A'\pi_n + B\pi_{n+1} = B\pi_n$$

provided we take $B = A'(1 - e^{-((1/2)\,e)})^{-1}$ (for $t = 0$, (17) holds for any $B \geqslant 0$). This completes the induction and shows that, with probability one, $\omega(t) \in U_n$, for some n, for all integer times. Moreover, since (17) holds for all n when $t \geqslant N = C(\log D_0 + \log K)$, this finishes the proof of the theorem for integer times. The remaining times follow from the corollary. ∎

3. ENSTROPHY BOUNDS

We prove a probabilistic analogue of the enstrophy balance:

Lemma 1. Given $\Phi(0)$, for any $t \in [0, 1]$,

$$\text{Prob}\{\Phi(t) \geqslant D^2\} \leqslant Ce^{-(c/R)(e^t D^2 - \Phi(0))}$$

Proof. Let $x(t) = 2\lambda(t)\,\Phi(t) = \lambda(t) \sum_{\mathbf{k}} |\omega_{\mathbf{k}}|^2$. Then by Ito's formula (recall that $\sum_{\mathbf{k}} \gamma_{\mathbf{k}} = 2R$ and thus that $\gamma_{\mathbf{k}} \leqslant 2R$, $\forall \mathbf{k}$):

$$\begin{aligned}
\frac{d}{dt} E[e^x] &= E\left[\left(\dot{\lambda}\lambda^{-1}x - 2\lambda \sum_{\mathbf{k}} \mathbf{k}^2 |\omega_{\mathbf{k}}|^2 + \lambda \sum_{\mathbf{k}} \gamma_{\mathbf{k}} + 2\lambda^2 \sum_{\mathbf{k}} \gamma_{\mathbf{k}} |\omega_{\mathbf{k}}|^2\right) e^x\right] \\
&\leqslant E[((\dot{\lambda}\lambda^{-1} - 2 + 4\lambda R)\, x + 2\lambda R)\, e^x]
\end{aligned}$$

where E denotes the expectation taken over the $f_{\mathbf{k}}$'s. We used the Navier–Stokes equation (3), $|\mathbf{k}| \geqslant 1$, and the fact that that the nonlinear term does not contribute. Take now $\lambda(t) = (1/8R)\, e^{(t-1)}$ so that $\dot{\lambda}\lambda^{-1} = 1$, $\dot{\lambda}\lambda^{-1} - 2 + 4\lambda R \leqslant -\frac{1}{2}$ and $2\lambda R \leqslant \frac{1}{4}$. So,

$$\frac{d}{dt} E[e^x] \leqslant E\left[\left(\frac{1}{4} - \frac{1}{2}x\right) e^x\right] \leqslant \frac{1}{2} - \frac{1}{4} E[e^x]$$

where the last inequality follows by using $(1-2x)\, e^x \leqslant 2 - e^x$. Thus, Gronwall's inequality implies that:

$$E[e^{x(t)}] \leqslant e^{-t/4} e^{x(0)} + 2 \leqslant 3e^{x(0)}$$

i,e.,

$$E\left[\exp\left(\frac{c}{R}\,\Phi(t)\, e^t\right)\right] \leqslant 3 \exp\left(\frac{c}{R}\,\Phi(0)\right)$$

with $c = e^{-1}/4$ which yields the claim by Chebycheff's inequality. ∎

This implies immediately the

Corollary. Let $D(t) \equiv e^{(-1/2)\,t}$ with $D^2 = \Phi(0)$, and let $t_1, \ldots, t_N \in [0, 1]$. Then

$$\text{Prob}\{\Phi(t_n) \leqslant \tfrac{3}{2} D(t_n)^2, \forall n = 1, \ldots, N\} \geqslant 1 - CNe^{-c(D^2/R)} \tag{18}$$

4. PROOF OF THE PROPOSITION

As usual, the stochastic equation (3) is defined by the integral equation,

$$\omega_{\mathbf{k}}(t) = e^{-t\mathbf{k}^2}\omega_{\mathbf{k}}(0) + \int_0^t ds\, e^{(s-t)\,\mathbf{k}^2} \sum_{\boldsymbol{l} \in \mathbf{Z}^2 \setminus \{\mathbf{0},\, \mathbf{k}\}} (\mathbf{k} \times \boldsymbol{l})\, |\boldsymbol{l}|^{-2}\, \omega_{\mathbf{k}-\boldsymbol{l}}(s)\, \omega_{\boldsymbol{l}}(s) + z_{\mathbf{k}}(t) \tag{19}$$

where $z_{\mathbf{k}}$ is an Ornstein–Uhlenbeck process, i.e., Gaussian with mean zero and covariance

$$Ez_{\mathbf{k}}(t)\, z_{\boldsymbol{l}}(s) = \delta_{\mathbf{k},\, -\boldsymbol{l}}\, \frac{1}{2\mathbf{k}^2}\, (e^{-(t-s)\,\mathbf{k}^2} - e^{-(t+s)\,\mathbf{k}^2})\, \gamma_{\mathbf{k}}$$

Our strategy to prove the proposition is the following. We fix a short timestep τ depending on D. By the corollary of the previous section the enstrophy can be assumed to satisfy the required bounds at discrete times $t_n = n\tau$. On the interval $[0, \tau]$ we prove an existence and uniqueness result for (19) in Lemma 3 by imposing a suitable condition on the smallness of the noise term z. At this point, the bound for $\|\omega(t)\|_{D(t)}$ will not improve as claimed in the proposition. However the enstrophy stays bounded and this information allows (Lemma 4) to improve the $\|\omega(t)\|_{D(t)}$-bound. Repeating Lemmas 3 and 4 on intervals $[t_n, t_{n+1}]$ the proposition follows.

Let

$$\tau = \delta D^{-4\alpha}$$

where δ will be chosen below (see after (29)), independently on D. We need the following standard result on the Ornstein–Uhlenbeck process:

Lemma 2. $\forall \mathbf{k} \in \mathbf{Z}^2$, $\mathrm{Prob}\{\sup_{t\in[0,\,\tau]} |z_{\mathbf{k}}(t)| \geqslant B\tau^{1/2}\} \leqslant Ce^{-(c/R)\, e^{|\mathbf{k}|} B^2}$.

This has the following simple consequence. Let $\mathscr{A}_D$ be the event

$$\{z \mid \forall \mathbf{k} \in \mathbf{Z}^2,\ \sup_{t\in[0,\,\tau]} |z_{\mathbf{k}}(t)| \leqslant \tau^{1/2} De^{-|\mathbf{k}|/4}\} \tag{20}$$

then Lemma 2 implies

$$\mathrm{Prob}\, \mathscr{A}_D \geqslant 1 - Ce^{-c(D^2/R)} \tag{21}$$

We now prove two lemmas. The first one, as we explained above, shows that the solution exists and that the solution satisfies the bounds of the proposition over a short time interval. For this, let Y_D be the Banach space equiped with the norm $\|\cdot\|_D$ and

$$X_D = \{\omega \in C^0([0, \tau], Y_D) \mid \|\omega\| \equiv \sup_{t \in [0, \tau]} \|\omega(t)\|_{D(t)} < \infty\} \tag{22}$$

where

$$D(t) = e^{-(1/2)\,t} D$$

Then we have,

Lemma 3. Let $z \in \mathscr{A}_D$ and suppose that $\|\omega(0)\|_D \leqslant D^\alpha$ and that $\Phi(0) \leqslant \frac{3}{2} D^2$. Then the solution exists in X_D and moreover,

$$\|\omega(t)\|_{\sqrt{2}\,D(t)} \leqslant (\sqrt{2}\,D(t))^\alpha, \qquad \Phi(t) \leqslant 2D(t)^2$$

for $t \in [0, \tau]$.

The second lemma improves on these bounds:

Lemma 4. Let $z \in \mathscr{A}_D$ and suppose that $\|\omega(0)\|_D \leqslant D^\alpha$ and that $\Phi(t) \leqslant 2D(t)^2$ for $t \in [0, \tau]$. Then $\|\omega(\tau)\|_{D(\tau)} \leqslant D(\tau)^\alpha$.

Proof of the Proposition. Let $t_n = n\tau$. By the corollary in the previous section (18), we may assume that $\Phi(t_n) \leqslant \frac{3}{2} D(t_n)^2$, for all $n = 1,..., N$, where $N = \delta^{-1} D^{4\alpha} - 1$ with probability

$$1 - CD^{4\alpha} e^{-c(D^2/R)} \tag{23}$$

We can thus repeat Lemmas 3 and 4 on intervals $[t_n, t_{n+1}]$, each time with probability (21). Hence, with probability bounded from below by (23) we deduce that $\|\omega(t)\|_{\sqrt{2}\,D(t)} \leqslant (\sqrt{2}\,D(t))^\alpha$ and $\Phi(t) \leqslant 2D(t)^2$ for all $t \in [0, 1]$, i.e., we have $\omega(t) \in U_{\sqrt{2e^{-t}}\,D}$ as required. By changing c and C, we can bound $D^{4\alpha}/R^{2\alpha}$ in (23) by the exponential, call a and A the new constants and obtain the claim of the proposition. ∎

Proof of Lemma 3. Write Eq. (19) as

$$\omega = F(\omega) \tag{24}$$

where

$$\begin{aligned} F_{\mathbf{k}}(v) &\equiv \omega^0_{\mathbf{k}}(t) + \int_0^t ds\, e^{(s-t)\,\mathbf{k}^2} \sum_{\boldsymbol{l} \in \mathbf{Z}^2 \setminus \{\mathbf{0}, \mathbf{k}\}} (\mathbf{k} \times \boldsymbol{l})\, |\boldsymbol{l}|^{-2}\, v_{\mathbf{k}-\boldsymbol{l}}(s)\, v_{\boldsymbol{l}}(s) \\ &\equiv \omega^0_{\mathbf{k}}(t) + N_{\mathbf{k}}(v)(t) \end{aligned} \tag{25}$$

and $\omega^0(t)$ equals:

$$\omega^0_{\mathbf{k}}(t) \equiv e^{-t\mathbf{k}^2}\omega_{\mathbf{k}}(0) + z_{\mathbf{k}}(t) \tag{26}$$

Using (20) and $z_{\mathbf{k}}(0) = 0$, which imply (trivially) that $\|z\| \leqslant D^\alpha$, and

$$e^{-t\mathbf{k}^2}e^{-D^{-\alpha}|\mathbf{k}|} \leqslant e^{-D(t)^{-\alpha}|\mathbf{k}|} \tag{27}$$

which holds for $t \in [0, \tau]$, we have

$$\|\omega^0\| \leqslant 2D^\alpha \tag{28}$$

We prove now that F is a contraction in the ball

$$B = \{v \in X_D : \|v - \omega^0\| \leqslant 1\} \tag{29}$$

provided the δ in $\tau = \delta D^{-4\alpha}$ is taken small enough (independently of D). To show that F maps B into itself, let $v \in B$. Then $\|v\| \leqslant 2D^\alpha + 1$, i.e.,

$$|v_{\mathbf{k}}(t)| \leqslant (2D^\alpha + 1)\, e^{-D(t)^{-\alpha}|\mathbf{k}|}\, |\mathbf{k}|^{-r} \tag{30}$$

We must prove that

$$|F_{\mathbf{k}}(v) - \omega^0_{\mathbf{k}}(t)| = |N_{\mathbf{k}}(v)(t)| \leqslant e^{-D(t)^{-\alpha}|\mathbf{k}|}\, |\mathbf{k}|^{-r} \tag{31}$$

$\forall \mathbf{k} \in \mathbf{Z}^2$ and $\forall t \in [0, \tau]$. Inserting (30) and $|\mathbf{k} \times \boldsymbol{l}|\, |\boldsymbol{l}|^{-2} \leqslant |\mathbf{k}|\, |\boldsymbol{l}|^{-1}$ in the second term of (25), we get:

$$|N_{\mathbf{k}}(v)(t)| \leqslant (2D^\alpha + 1)^2 \int_0^t ds\, e^{(s-t)\mathbf{k}^2} \sum_{\boldsymbol{l} \in \mathbf{Z}^2 \setminus \{\mathbf{0}, \mathbf{k}\}} e^{-D(s)^{-\alpha}|\mathbf{k}-\boldsymbol{l}|} e^{-D(s)^{-\alpha}|\boldsymbol{l}|} \times |\mathbf{k}-\boldsymbol{l}|^{-r}\, |\boldsymbol{l}|^{-r-1}\, |\mathbf{k}| \tag{32}$$

Then, using the bound

$$\sum_{\boldsymbol{l} \in \mathbf{Z}^2 \setminus \{\mathbf{0}, \mathbf{k}\}} |\mathbf{k}-\boldsymbol{l}|^{-r}\, |\boldsymbol{l}|^{-r-1} \leqslant C\, |\mathbf{k}|^{-r} \tag{33}$$

(since $r > 1$), the triangle inequality $-|\mathbf{k}-\boldsymbol{l}| - |\boldsymbol{l}| \leqslant -|\mathbf{k}|$ and

$$\tfrac{1}{2}(s-t)\,\mathbf{k}^2 \leqslant (e^{(1/2)\alpha s} - e^{(1/2)\alpha t})\, |\mathbf{k}|\, D^{-\alpha} = (D(s)^{-\alpha} - D(t)^{-\alpha})\, |\mathbf{k}| \tag{34}$$

which holds for $0 \leqslant s \leqslant t \leqslant 1$ and D large enough, one gets that

$$|N_{\mathbf{k}}(v)(t)| \leqslant (2D^{\alpha}+1)^2 \, |\mathbf{k}| \, C \, |\mathbf{k}|^{-r} \, e^{-D(t)^{-\alpha} |\mathbf{k}|} \int_0^t ds \, e^{(1/2)(s-t) \, \mathbf{k}^2}$$
$$= (2D^{\alpha}+1)^2 \, C \, |\mathbf{k}|^{-r} \, e^{-D(t)^{-\alpha} |\mathbf{k}|} 2 \, |\mathbf{k}|^{-1} \, (1-e^{-(1/2) \, t\mathbf{k}^2}) \tag{35}$$

Since $|\mathbf{k}|^{-1} \, (1-e^{-(1/2) \, t\mathbf{k}^2}) \leqslant t^{1/2} \leqslant \delta^{1/2} D^{-2\alpha}$ (31) follows for δ small enough (but independent of D). The contractive property is proven similarly.

Combining the fact that the solution is contained in the ball (29) and the inequality $2D^{\alpha}+1 \leqslant (\sqrt{2} \, e^{-t/2} D)^{\alpha} = (\sqrt{2} \, D(t))^{\alpha}$ (which holds, since $\alpha > 2$, for $t \in [0, \tau]$ and D large enough) we obtain,

$$\|\omega(t)\|_{\sqrt{2} \, D(t)} \leqslant \|\omega(t)\|_{D(t)} \leqslant (\sqrt{2} \, D(t))^{\alpha} \tag{36}$$

for $t \in [0, \tau]$ and D large enough.

To conclude we need to prove that $\Phi(t) \leqslant 2D(t)^2$. By (24), (25),

$$\Phi(t) = \tfrac{1}{2} \, \|\omega(t)\|_2^2 \leqslant \tfrac{1}{2} (\|\omega^0(t)\|_2 + \|N(\omega)(t)\|_2)^2 \tag{37}$$

By (20) and $z_{\mathbf{k}}(0) = 0$, the L^2-norm of $z(t)$ is bounded by $C\tau^{1/2} D = C\delta^{1/2} D^{1-2\alpha}$, the L^2-norm of the first term in (26) is bounded by $\|\omega(0)\| \leqslant \sqrt{2\Phi(0)} \leqslant \sqrt{3} \, D$ and, using (31), the L^2-norm of $N(\omega)(t)$ is bounded by $(\sum_{\mathbf{k} \in \mathbf{Z}^2 \setminus \{0\}} |\mathbf{k}|^{-2r})^{1/2} = C$ (since $r > 1$). Thus, we obtain the claim provided D is large enough. ∎

Proof of Lemma 4. We note first that $\Phi(\tau) \leqslant 2D(\tau)^2$ implies

$$|\omega_{\mathbf{k}}(\tau)| \leqslant \sqrt{2} \, D(\tau) \leqslant D(\tau)^{\alpha} \, e^{-D(\tau)^{-\alpha} |\mathbf{k}|} \, |\mathbf{k}|^{-r} \tag{38}$$

provided $|\mathbf{k}| \leqslant D^{\beta}$, $\alpha > 1 + r\beta$ and D is large enough. Hence, we only need to consider $|\mathbf{k}| > D^{\beta}$. Below, we take as β any number strictly larger than 1.

We can now conclude the proof of the lemma by using the following bound on the nonlinear term of the Navier–Stokes equation, which improves (35) for $\mathbf{k}$ large enough:

Lemma 5. $\forall \mathbf{k}$ such that $|\mathbf{k}| \geqslant D^{\beta}$, and $\forall t \in [0, \tau]$,

$$|N_{\mathbf{k}}(\omega)(t)| \leqslant c(1-e^{-(1/2) \, t\mathbf{k}^2}) \, D^{\alpha} e^{-D(t)^{-\alpha} |\mathbf{k}|} \, |\mathbf{k}|^{-r} \tag{39}$$

where c can be taken small if D is large enough.

Returning to the proof of Lemma 4, we have to prove the following bound:

$$|\omega_{\mathbf{k}}(\tau)| \leqslant e^{-(1/2) \, \alpha\tau} \, D^{\alpha} e^{-D(\tau)^{-\alpha} |\mathbf{k}|} \, |\mathbf{k}|^{-r} \tag{40}$$

We shall use (25), (26) and bound each term. Using $\|\omega(0)\|_D \leqslant D^\alpha$ and

$$e^{-(1/2)\tau \mathbf{k}^2} e^{-D^{-\alpha} |\mathbf{k}|} \leqslant e^{-D(\tau)^{-\alpha} |\mathbf{k}|} \tag{41}$$

which is similar to (27), we get,

$$|e^{-\tau \mathbf{k}^2} \omega_{\mathbf{k}}(0)| \leqslant e^{-(1/2)\tau \mathbf{k}^2} D^\alpha e^{-D(\tau)^{-\alpha} |\mathbf{k}|} |\mathbf{k}|^{-r} \tag{42}$$

For $z_{\mathbf{k}}(\tau)$, use (20), $z_{\mathbf{k}}(0) = 0$, and

$$e^{-|\mathbf{k}|/8} \leqslant e^{-D(\tau)^{-\alpha} |\mathbf{k}|} |\mathbf{k}|^{-r} \tag{43}$$

for $|\mathbf{k}|$ large, to get

$$|z_{\mathbf{k}}(\tau)| \leqslant \tau^{1/2} D e^{-|\mathbf{k}|/8} e^{-D(\tau)^{-\alpha} |\mathbf{k}|} |\mathbf{k}|^{-r} \tag{44}$$

Finally, we use (39) to bound $N_{\mathbf{k}}(\omega)(\tau)$. Combining (42), (44), (39), we obtain (40) using

$$e^{-(1/2)\tau \mathbf{k}^2} + c(1 - e^{-(1/2)\tau \mathbf{k}^2}) + \tau^{1/2} D^{1-\alpha} e^{-|\mathbf{k}|/8} \leqslant e^{-(1/2)\alpha\tau} \tag{45}$$

Since $\tau = \delta D^{-4\alpha}$, this last estimate holds for c small, $|\mathbf{k}| \geqslant D^\beta$, and D large enough.

Proof of Lemma 5. Consider first the case $D^\beta \leqslant |\mathbf{k}| \leqslant A D^\alpha$, where A is a large enough constant (chosen below). We bound $|\mathbf{k} \times \boldsymbol{l}|\, |\boldsymbol{l}|^{-2} \leqslant |\mathbf{k}|\, |\boldsymbol{l}|^{-1}$ and split the sum in (25) into

$$\Bigg(\sum_{\mathbf{0} \neq |\boldsymbol{l}| \leqslant (|\mathbf{k}|/2)} + \sum_{\boldsymbol{l} \neq \mathbf{k},\, |\boldsymbol{l}| > (|\mathbf{k}|/2)} \Bigg) |\omega_{\mathbf{k}-\boldsymbol{l}}(s)|\, |\omega_{\boldsymbol{l}}(s)|\, |\mathbf{k}|\, |\boldsymbol{l}|^{-1} \equiv \Sigma_1 + \Sigma_2 \tag{46}$$

In the first sum, we bound, using Lemma 3,

$$|\omega_{\mathbf{k}-\boldsymbol{l}}(s)| \leqslant C D^\alpha |\mathbf{k} - \boldsymbol{l}|^{-r} \leqslant C D^\alpha |\mathbf{k}|^{-r}$$

since $|\mathbf{k} - \boldsymbol{l}| \geqslant \frac{1}{2} |\mathbf{k}|$. Also, from Lemma 3

$$\|\omega(s)\|_2 = \sqrt{2}\, \Phi(s)^{1/2} \leqslant 2D(s) \tag{47}$$

so Schwartz' inequality yields

$$\begin{aligned} \sum_{\mathbf{0} \neq |\boldsymbol{l}| \leqslant (|\mathbf{k}|/2)} |\omega_{\boldsymbol{l}}(s)|\, |\boldsymbol{l}|^{-1} &\leqslant 2D(s) \Bigg(\sum_{\mathbf{0} \neq |\boldsymbol{l}| \leqslant (|\mathbf{k}|/2)} |\boldsymbol{l}|^{-2} \Bigg)^{1/2} \\ &\leqslant C D (\log |\mathbf{k}|)^{1/2} \end{aligned} \tag{48}$$

Combining these two bounds we get

$$\Sigma_1 \leqslant CD\, |\mathbf{k}|\, (\log |\mathbf{k}|)^{1/2}\, D^\alpha\, |\mathbf{k}|^{-r} \tag{49}$$

For the second sum, we use $|\omega_{\boldsymbol{l}}(s)| \leqslant CD^\alpha\, |\boldsymbol{l}|^{-r}$, together with (47) and Schwartz' inequality to bound it by

$$\Sigma_2 \leqslant CD\, |\mathbf{k}|\, D^\alpha \left(\sum_{\boldsymbol{l} \neq \mathbf{k},\ |\boldsymbol{l}| > (|\mathbf{k}|/2)} |\boldsymbol{l}|^{-2(r+1)} \right)^{1/2} \leqslant CD\, |\mathbf{k}|\, D^\alpha\, |\mathbf{k}|^{-r} \tag{50}$$

Inserting (49) and (50) to $N_{\mathbf{k}}(\omega)(t)$ and performing the integral over time we get the bound

$$\begin{aligned} |N_{\mathbf{k}}(\omega)(t)| &\leqslant CD\, |\mathbf{k}|^{-1}\, (\log |\mathbf{k}|)^{1/2}\, (1-e^{-t\mathbf{k}^2})\, D^\alpha\, |\mathbf{k}|^{-r} \\ &\leqslant Ce^{CA} D^{1-\beta} (\log D)^{1/2}\, (1-e^{-(1/2)\, t\mathbf{k}^2})\, D^\alpha e^{-D(t)^{-\alpha}\, |\mathbf{k}|}\, |\mathbf{k}|^{-r} \end{aligned} \tag{51}$$

where we used $D^\beta \leqslant |\mathbf{k}| \leqslant AD^\alpha$ and

$$1 \leqslant e^{-D(t)^{-\alpha}\, |\mathbf{k}|} e^{CA}$$

which holds since $|\mathbf{k}| \leqslant AD^\alpha$. The claim of the lemma follows, for $D^\beta \leqslant |\mathbf{k}| \leqslant AD^\alpha$, since D is assumed to be large enough and we choose $\beta > 1$.

Consider now the case $|\mathbf{k}| > AD^\alpha$. Using the bound (35), we get

$$\begin{aligned} |N_{\mathbf{k}}(\omega)(t)| &\leqslant C\, |\mathbf{k}|^{-1}\, (1-e^{-(1/2)\, t\mathbf{k}^2})\, D^{2\alpha} e^{-D(t)^{-\alpha}\, |\mathbf{k}|}\, |\mathbf{k}|^{-r} \\ &\leqslant c(1-e^{-(1/2)\, t\mathbf{k}^2})\, D^\alpha e^{-D(t)^{-\alpha}\, |\mathbf{k}|}\, |\mathbf{k}|^{-r} \end{aligned} \tag{52}$$

by choosing A large enough (thus, we first choose A large so that (52) holds with c small enough for (45) to be true and then we choose D large so that the RHS of (51) is bounded by the RHS of (39) with c small enough). ∎

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